NAME ______________________ DATE ____________ PERIOD ____________

8.3 Dot Products and Vector Projections

KeyConcept Dot Product of Vectors in a Plane

The dot product of $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ is defined as $\boxed{\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2.}$

Notice that unlike vector addition and scalar multiplication, the dot product of two vectors yields a scalar, not a vector. As demonstrated above, two nonzero vectors are perpendicular if and only if their dot product is 0. Two vectors with a dot product of 0 are said to be **orthogonal**.

KeyConcept Orthogonal Vectors

The vectors a and b are orthogonal if and only if $\boxed{\mathbf{a} \cdot \mathbf{b} = 0.}$

The terms *perpendicular* and *orthogonal* have essentially the same meaning, except when **a** or **b** is the zero vector. The zero vector is orthogonal to any vector **a**, since $\langle 0, 0 \rangle \cdot \langle a_1, a_2 \rangle = 0a_1 + 0a_2$ or 0. However, since the zero vector has no magnitude or direction, it cannot be perpendicular to **a**.

Finding the Dot Product to Determine Orthogonal Vectors:

Examples: *Find the dot product of* **u** *and* **v**. *Then determine if* **u** *and* **v** *are orthogonal.*

1. $\mathbf{u} = \langle 3, 6 \rangle, \mathbf{v} = \langle -4, 2 \rangle$

$u \cdot v = \langle 3,6 \rangle \cdot \langle -4,2 \rangle$

$u \cdot v = 3 \cdot (-4) + 6 \cdot (2)$

$u \cdot v = -12 + 12$

$\boxed{u \cdot v = 0}$, therefore, u and v are $\boxed{\text{orthogonal}}$

2. $\mathbf{u} = 2\mathbf{i} + 5\mathbf{j}, \mathbf{v} = 8\mathbf{i} + 4\mathbf{j}$

$u \cdot v = 2(8) + 5 \cdot (4)$

$u \cdot v = 16 + 20$

$\boxed{u \cdot v = 36}$

u and v are $\boxed{\text{not orthogonal}}$

KeyConcept Properties of the Dot Product

If u, v, and w are vectors and *k* is a scalar, then the following properties hold.

Commutative Property	$\boxed{\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}}$
Distributive Property	$\boxed{\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}}$
Scalar Multiplication Property	$\boxed{k(\mathbf{u} \cdot \mathbf{v}) = k\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot k\mathbf{v}}$
Zero Vector Dot Product Property	$\boxed{\mathbf{0} \cdot \mathbf{u} = 0}$
Dot Product and Vector Magnitude Relationship	$\boxed{\mathbf{u} \cdot \mathbf{u} = \lvert \mathbf{u} \rvert^2}$

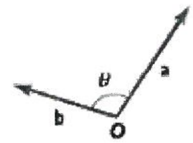
Using the Dot Product to Find Magnitude:

Examples: *Use the dot product to find the magnitude of the given vector.*

1. $\mathbf{b} = \langle 12, 16\rangle$

2. $\mathbf{c} = \langle 2, -7\rangle$

The angle θ between any two nonzero vectors **a** and **b** is the corresponding angle between these vectors when placed in standard position, as shown. This angle is always measured such that $0 \le \theta \le \pi$ or $0^\circ \le \theta \le 180^\circ$. The dot product can be used to find the angle between two nonzero vectors.



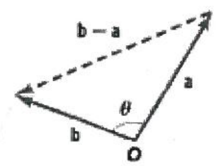
StudyTip

Parallel and Perpendicular Vectors Two vectors are perpendicular if the angle between them is 90°. Two vectors are parallel if the angle between them is 0° or 180°.

KeyConcept Angle Between Two Vectors

If θ is the angle between nonzero vectors **a** and **b**, then

$$\boxed{\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}}.$$



Finding the Angle Between Two Vectors:

Examples: *Find the angle θ between* **u** *and* **v** *to the nearest tenth of a degree.*

1. $\mathbf{u} = \langle 6, 2\rangle, \mathbf{v} = \langle -4, 3\rangle$

2. $\mathbf{u} = 9\mathbf{i} + 5\mathbf{j}, \mathbf{v} = -6\mathbf{i} + 7\mathbf{j}$

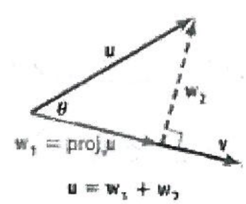
StudyTip

Perpendicular Component The vector $\mathbf{w}_2$ is called the *component of* **u** *perpendicular to* **v**.

KeyConcept Projection of u onto v

Let **u** and **v** be nonzero vectors, and let $\mathbf{w}_1$ and $\mathbf{w}_2$ be vector components of **u** such that $\mathbf{w}_1$ is parallel to **v** as shown. Then vector $\mathbf{w}_1$ is called the **vector projection** of **u** onto **v**, denoted $\text{proj}_\mathbf{v}\mathbf{u}$, and

$$\boxed{\text{proj}_\mathbf{v}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}.}$$



Finding the Projection of u onto v:

Examples: *Find the projection of* **u** *onto* **v** *given the vectors. Then write u as the sum of two orthogonal vectors, one of which is the projection of* **u** *onto* **v**.

1. $\mathbf{u} = \langle 3, 2\rangle, \mathbf{v} = \langle 5, -5\rangle$

Step 1: Find the projection of u onto v

$$\text{proj}_v u = \left(\frac{u\cdot v}{|v|^2}\right)v = \left(\frac{3(5)+2(-5)}{(\sqrt{50})^2}\right)\langle 5,-5\rangle = \frac{5}{50}\langle 5,-5\rangle = \boxed{\langle \tfrac{1}{2}, -\tfrac{1}{2}\rangle}$$

Step 2: Find vector w_2

Since $\boxed{u = w_1 + w_2} \rightarrow w_2 = u - w_1$, $\boxed{w_1 = \text{proj}_v u}$

$$w_2 = u - \text{proj}_v u$$

$$w_2 = \langle 3, 2\rangle - \langle \tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$= \boxed{\langle \tfrac{5}{2}, \tfrac{5}{2}\rangle}$$

Then, $\boxed{u = \langle \tfrac{1}{2}, -\tfrac{1}{2}\rangle + \langle \tfrac{5}{2}, \tfrac{5}{2}\rangle}$

2. $\mathbf{u} = \langle 1,2 \rangle, \mathbf{v} = \langle 8,5 \rangle$

$\text{Proj}_v u = \left(\frac{1(8)+2(5)}{(\sqrt{89})^2}\right)\langle 8,5\rangle = \frac{18}{89}\langle 8,5\rangle = \boxed{\langle \frac{144}{89}, \frac{90}{89} \rangle}$

$w_2 = \langle 1,2\rangle - \langle \frac{144}{89}, \frac{90}{89}\rangle$

$= \langle -\frac{55}{89}, \frac{88}{89}\rangle$

$= \boxed{\langle -\frac{55}{89}, \frac{88}{89}\rangle}$

$\boxed{u = \langle \frac{144}{89}, \frac{90}{89}\rangle + \langle -\frac{55}{89}, \frac{88}{89}\rangle}$

3. $\mathbf{u} = \langle 4,-3 \rangle, \mathbf{v} = \langle 2,6 \rangle$

$\text{Proj}_v u = \left(\frac{4(2)+(-3)(6)}{(\sqrt{40})^2}\right)\langle 2,6\rangle$

$\text{Proj}_v u = \frac{-10}{40}\langle 2,6\rangle = \boxed{\langle -\frac{1}{2}, \frac{-3}{2}\rangle}$

$w_2 = \langle 4,-3\rangle - \langle -\frac{1}{2}, -\frac{3}{2}\rangle$

$\boxed{= \langle \frac{9}{2}, \frac{-3}{2}\rangle}$

$\boxed{u = \langle -\frac{1}{2}, -\frac{3}{2}\rangle + \langle \frac{9}{2}, \frac{-3}{2}\rangle}$

4. $\mathbf{u} = \langle -3,4 \rangle, \mathbf{v} = \langle 6,1 \rangle$

$\boxed{\text{Proj}_v u = \langle -\frac{84}{37}, \frac{-14}{37}\rangle}$

$\boxed{u = \langle -\frac{84}{37}, \frac{-14}{37}\rangle + \langle -\frac{27}{37}, \frac{162}{37}\rangle}$

Using the Dot product to find Magnitude:

1. $b = \langle 12, 16 \rangle$

$|b| = \sqrt{b \cdot b}$

$|b| = \sqrt{12 \cdot 12 + 16 \cdot 16}$

$|b| = \sqrt{400}$

$|b| = 20$

2. $c = \langle 2, -7 \rangle$

$|c| = \sqrt{c \cdot c}$

$|c| = \sqrt{2^2 + (-7)^2}$

$|c| = \sqrt{4 + 49}$

$|c| = \sqrt{53}$

Finding the Angle Between two Vectors:

1. $u = \langle 6, 2 \rangle$, $v = \langle -4, 3 \rangle$

$$\cos\theta = \frac{u \cdot v}{|u| \cdot |v|} = \frac{6 \cdot (-4) + 2(3)}{(\sqrt{6^2+2^2})(\sqrt{(-4)^2+3^2})} = \frac{-18}{(\sqrt{40})(\sqrt{25})} = \frac{-18}{\sqrt{1000}}$$

$$\theta = \cos^{-1}\left(\frac{-18}{\sqrt{1000}}\right)$$

$$\boxed{\theta = 124.7^\circ}$$

2. $u = 9i + 5j$, $v = -6i + 7j$

$$\cos\theta = \frac{9(-6) + 5(7)}{(\sqrt{9^2+5^2})(\sqrt{(-6)^2+7^2})} = \frac{-19}{(\sqrt{106})(\sqrt{85})} = \frac{-19}{\sqrt{9010}}$$

$$\theta = \cos^{-1}\left(\frac{-19}{\sqrt{9010}}\right)$$

$$\boxed{\theta = 101.5^\circ}$$