Name______________________________ Date____________ Period________

Chapter 6 Matrices Study Guide

Matrix: A rectangular array of numbers

Dimensions (order) of a matrix is represented "rows x columns"

Examples: What are the dimensions of following matrices?

$C = \begin{bmatrix} 2 & 5 \\ 1 & -8 \end{bmatrix}$ $A = \begin{bmatrix} 4 & 8 \\ 7 & -1 \\ -3 & 5 \end{bmatrix}$ $E = \begin{bmatrix} 17 & 22 & 17 & 16 \\ 23 & 30 & 24 & 19 \end{bmatrix}$ $Q = \begin{bmatrix} 3 & \pi & -4 \end{bmatrix}$ $M = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}$

$C_{2\times2}$ $A_{3\times2}$ $E_{2\times4}$ $Q_{1\times3}$ $M_{3\times1}$

Matrix __M__ is also called to be **a column vector** since it has only one column.

****When the number of rows = number of columns, the matrix is called a square matrix.**

Matrices __A[illegible]d C__ are said to be square matrices.

<u>Basic Matrix Operations</u>

Matrix Addition: Two matrices having the same dimensions can be added to produce a new matrix. Only the corresponding elements are added. If two matrices do not have the same order, then, they cannot be added.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$

Example: Find M + N.

$M = \begin{bmatrix} 2 & -8 & 14 \\ 5 & 10 & 6 \end{bmatrix}$ $N = \begin{bmatrix} -5 & 9 & 8 \\ 13 & -4 & 1 \end{bmatrix}$

$$M+N = \begin{bmatrix} -3 & 1 & 22 \\ 18 & 6 & 7 \end{bmatrix}$$

Matrix Subtraction: We add the additive inverse of the matrix to subtract a matrix from another matrix with the same dimensions.

Example: Multiply each element on matrix C by -1, which is also called additive inverse (-C)

$C = \begin{bmatrix} 2 & 5 \\ 1 & -8 \end{bmatrix}$ $-C = \begin{bmatrix} -2 & -5 \\ -1 & 8 \end{bmatrix}$

Example: Find W – X.

$W = \begin{bmatrix} 1 & -5 \\ -3 & 1 \\ 4 & 2 \end{bmatrix}$ $-X = \begin{bmatrix} -3 & -2 \\ 0 & -7 \\ -8 & +6 \end{bmatrix}$ $W-X = \begin{bmatrix} -2 & -7 \\ -3 & -6 \\ -4 & 8 \end{bmatrix}$

Scalar Multiplication: When we multiply a matrix by a real number (known as a scalar), the resulting matrix (new matrix) is produced by multiplying each element in the original matrix by the given scalar.

$$k\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{bmatrix}$$

Example: Find -8A.

$$A = \begin{bmatrix} 4 & 8 \\ 7 & -1 \\ -3 & 5 \end{bmatrix}$$

$$-8A = -8\begin{bmatrix} 4 & 8 \\ 7 & -1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -32 & -64 \\ -56 & +8 \\ +24 & -40 \end{bmatrix}$$

Transpose: Taking the transpose of a matrix is basically interchanging the rows and columns of a matrix. Denoted by a superscript, "T" or "t".

Example 1: Find transpose

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -3 \\ 3 & 5 & 2 \end{bmatrix} = A$$

$$A^T = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 5 \\ 0 & -3 & 2 \end{bmatrix}$$

Example 2: Find transpose

$$N = \begin{bmatrix} -5 & 9 & 8 \\ 13 & -4 & 1 \end{bmatrix}$$

$$N^T = \begin{bmatrix} -5 & 13 \\ 9 & -4 \\ 8 & 1 \end{bmatrix}$$

Matrix Multiplication: Two matrices (A and B) can only be multiplied if the number of columns of A equals the number of rows of B.

matrix A	$\cdot$	matrix B	$=$	AB
3×2		2×4		3×4

Equal

Dimensions of AB

Example: Find the matrix product for the following matrices, A and B.

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & -5 & 2 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 0 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} c & d \\ e & f \end{bmatrix} = \begin{bmatrix} -3 & 10 \\ -7 & -16 \end{bmatrix}$$

*** Notice that the dimensions of the first matrix (A) are 2X3 and the dimensions of the second matrix (B) are 3X2. The dimensions of their product will be 2X2.

c = the sum of Row 1 of A times the first column B

c = 1.(-3) + 2.(0) + 0.(1) = -3

e = the sum of Row 2 of A times the first column of B

e = 3(-3)+(-5).(0)+2.(1) = -7

d = the sum of Row 1 times the second column of B

d = 1.(2)+2.(4)+0.(-1) = 10

f = the sum of Row 2 of A times the second column of B

f = 3.(2)+(-5).4+2.(-1) = -16

Examples: find each matrix product if it is defined.

a. $\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}4 & 2\\ 5 & 6\end{bmatrix} = \begin{bmatrix}1\times4+1\times5 & 1\times2+1\times6\\ 2\times4+3\times5 & 2\times2+3\times6\end{bmatrix} = \begin{bmatrix}9 & 8\\ 23 & 22\end{bmatrix}$

$2\times2 \quad 2\times2$

b. $\begin{bmatrix}9 & -4 & 4\\ 2 & -1 & -6\end{bmatrix}\begin{bmatrix}2 & -1 & 0\\ 0 & 1 & -3\\ 3 & 5 & 2\end{bmatrix} = \begin{bmatrix}9\times2+(-4)\times0+4\cdot(3) & 9\cdot(-1)+(-4)\cdot1+(-4)\cdot5 & 9\cdot0+(-4)\cdot(-3)+4\cdot2\\ 2\cdot2+(-1)\cdot0+(-6)\cdot3 & 2(-1)+(-1)\cdot1+(-6)5 & 2\cdot0+(-1)(-3)+(-6)\cdot2\end{bmatrix}$

$2\times3 \quad 3\times3$

$= \begin{bmatrix}30 & -33 & 20\\ -14 & -33 & -9\end{bmatrix}$

c. $\begin{bmatrix}8 & -10\\ 0 & 3\\ 6 & 4\end{bmatrix}\begin{bmatrix}-2\\ -9\\ 1\end{bmatrix}$ = undefined

$3\times2 \quad 3\times1$

not same, so, matrix product is not defined.

Example: find AB and BA if possible.

$A=\begin{bmatrix}1 & -5 & 4\\ -2 & 3 & 5\\ 6 & -3 & 1\end{bmatrix}$, $B=[2 \quad -1 \quad -8]$

$3\times3 \qquad 1\times3$

$BA = \begin{bmatrix}2 & -1 & -8\end{bmatrix}\begin{bmatrix}1 & -5 & 4\\ -2 & 3 & 5\\ 6 & -3 & 1\end{bmatrix}$ possible

$1\times3 \quad 3\times3$

We will get 1×3

$BA = [2\cdot1+(-1)(-2)+(-8)\cdot6 \quad 2\cdot(-5)+(-1)3+(-1)(-3) \quad 2\cdot4+(-1)\cdot5+(-8)\cdot1]$

$BA = [-44 \quad -10 \quad -5]$

$AB = [\quad][\quad]$

$3\times3 \quad 1\times3$

not same

Product AB is not possible

Determinants: Determinant is a square array of numbers or variables enclosed between two parallel lines. Every square matrix has a number associated with it called its determinant.

Second Order Determinant (for a 2X2 matrix): The difference of the products of the two diagonals

$$\begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc$$

Example 1: Find determinant

$\begin{vmatrix}-2 & 5\\ 6 & 8\end{vmatrix}$

det $= (-2)\cdot8 - 6\cdot5$

det $= -16 - 30$

$= -46$

Example 2: Find determinant

$\begin{vmatrix}7 & 4\\ -3 & 2\end{vmatrix} = 7\cdot2-(-3)\cdot4$

$= 14+12$

$= 26$

Third-Order Determinant (for a 3X3 matrix): Expansion of minors can be used to evaluate this determinant. Minor of an element is the determinant formed by deleting the row and column of containing element.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Example: Find determinant

$$\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix} = 2\begin{vmatrix} 5 & -4 \\ 9 & 0 \end{vmatrix} - 7\begin{vmatrix} -1 & -4 \\ 6 & 0 \end{vmatrix} + (-3)\begin{vmatrix} -1 & 5 \\ 6 & 9 \end{vmatrix}$$

$$= 2[5\cdot 0-(-4)\cdot 9]-7[(-1)\cdot 0-(-4)\cdot 6]-3[(-1)\cdot 9-5\cdot 6]$$

$$= 72+168+117 = 357$$

Identity Matrix: It is a matrix used in matrix multiplication such that when multiplied by another matrix, the resulting matrix is equal to the original matrix. You may call it as non-effective matrix.

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

***Therefore, **IA = A and AI = A**

Inverse Matrix: Two matrices are inverses of each other, if their product is the identity matrix.

if $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where $ad\text{-}bc \neq 0$

***Therefore, $\boldsymbol{AA^{-1} = I \text{ or } A^{-1}A = I}$

Example: Find the inverse of matrix X and matrix Y

$X = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}$ $\qquad$ $Y = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}$

$$X^{-1} = \frac{1}{2\cdot 4-(-1)\cdot 2}\begin{bmatrix} 4 & -2 \\ +1 & 2 \end{bmatrix}$$

$$X^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4/10 & -2/10 \\ 1/10 & 2/10 \end{bmatrix} = \begin{bmatrix} 2/5 & -1/5 \\ 1/10 & 1/5 \end{bmatrix}$$

***Are two matrices inverses of each other?**

If the product of two matrices is equal to the Identity Matrix, then, they are inverses of each other.

Example: Determine whether matrices P and Q are inverses of each other.

$P = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ *and* $Q = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

$$P\cdot Q = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 3\cdot 1+4\cdot(-1/2) & 3\cdot(-2)+4\cdot(3/2) \\ 1\cdot 1+2\cdot(-1/2) & 1\times(-2)+2(3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since the product of P & Q equals the identity matrix,

P and Q are inverses.

Solving System of Linear Equations Using Inverses:

Invertible Square Linear Systems: $\boxed{\text{If } AX=B \text{ and } A \text{ is invertible, then } X = A^{-1}B}$

Use Inverse Matrix to solve each system. If the system cannot be solved using inverse matrix, then, write "impossible"

Example 1:

$x - 3y = 5$
$-3x + 6y = 8$

$$\underbrace{\begin{bmatrix} 1 & -3 \\ -3 & 6 \end{bmatrix}}_{A}\overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{B}{\begin{bmatrix} 5 \\ 8 \end{bmatrix}}$$

Inverse

$$\frac{1}{1\cdot 6-(-3)(-3)}\begin{bmatrix} 6 & 3 \\ 3 & 1 \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} 6 & 3 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 \\ -1 & -1/3 \end{bmatrix}$$

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -1/3 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} (-2)\cdot 5 + (-1)\cdot 8 \\ (-1)\cdot 5 + (-1/3)\cdot 8 \end{bmatrix}$$

$$= \begin{bmatrix} -10-8 \\ -5-8/3 \end{bmatrix}$$

$$= \begin{bmatrix} -18 \\ -\frac{23}{3} \end{bmatrix}$$

$x = -18$
$y = -\frac{23}{3}$

Example 2:

$2a - 3b = -8$
$4a + 3b = -34$

$$\underbrace{\begin{bmatrix} 2 & -3 \\ 4 & 3 \end{bmatrix}}_{A}\overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{B}{\begin{bmatrix} -8 \\ -34 \end{bmatrix}}$$

Inverse

$$\frac{1}{2\cdot 3-(-3)\cdot 4}\begin{bmatrix} 3 & 3 \\ -4 & 2 \end{bmatrix} = \frac{1}{6+12}\begin{bmatrix} 3 & 3 \\ -4 & 2 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 3 & 3 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 & 1/6 \\ -2/9 & 1/9 \end{bmatrix}$$

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/6 & 1/6 \\ -2/9 & 1/9 \end{bmatrix}\begin{bmatrix} -8 \\ -34 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6}(-8)+\frac{1}{6}(-34) \\ -\frac{2}{9}\cdot(-8)+\frac{1}{9}(-34) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-8}{6}+\frac{(-34)}{6} \\ \frac{16}{9}+\frac{(-34)}{9} \end{bmatrix}$$

$$= \begin{bmatrix} -42/6 \\ -18/9 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ -2 \end{bmatrix}$$

$x = -7$
$y = -2$