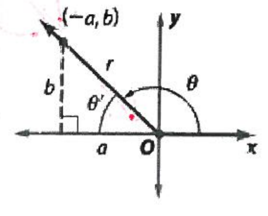
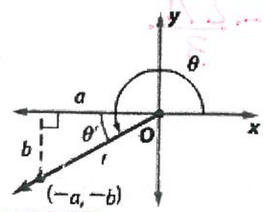
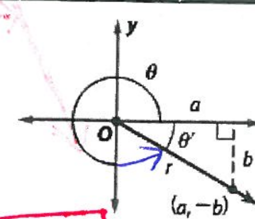
Reference Angles:

In order to find the values of trig functions of angles that are neither acute nor quadrantal, consider 3 cases shown below in which *a* and *b* are positive real numbers.

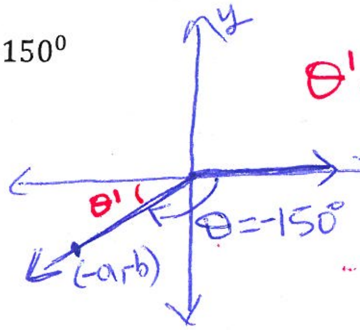
> **StudyTip**
> **Reference Angles** Notice that in some cases, the three trigonometric values of θ and θ' (read *theta prime*) are the same. In other cases, they differ only in sign. *

Quadrant II		Quadrant III		Quadrant IV	
					
$\sin\theta = \frac{b}{r}$	$\sin\theta' = \frac{b}{r}$	* $\boxed{\sin\theta = -\frac{b}{r}}$	$\sin\theta' = \frac{b}{r}$	* $\boxed{\sin\theta = -\frac{b}{r}}$	$\sin\theta' = \frac{b}{r}$
* $\boxed{\cos\theta = -\frac{a}{r}}$	$\cos\theta' = \frac{a}{r}$	* $\boxed{\cos\theta = -\frac{a}{r}}$	$\cos\theta' = \frac{a}{r}$	$\cos\theta = \frac{a}{r}$	$\cos\theta' = \frac{a}{r}$
$\tan\theta = -\frac{b}{a}$	$\tan\theta' = \frac{b}{a}$	$\tan\theta = \frac{b}{a}$	$\tan\theta' = \frac{b}{a}$	* $\boxed{\tan\theta = -\frac{b}{a}}$	$\tan\theta' = \frac{b}{a}$

This angle θ', called a **reference angle**, can be used to find the trigonometric values of any angle θ.

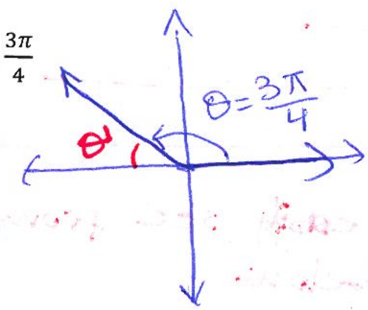
Examples: Sketch each angle and find its reference angle.

1. -150^0



$\theta' = 180° - 150°$
$= 30°$

2. $\frac{3\pi}{4}$



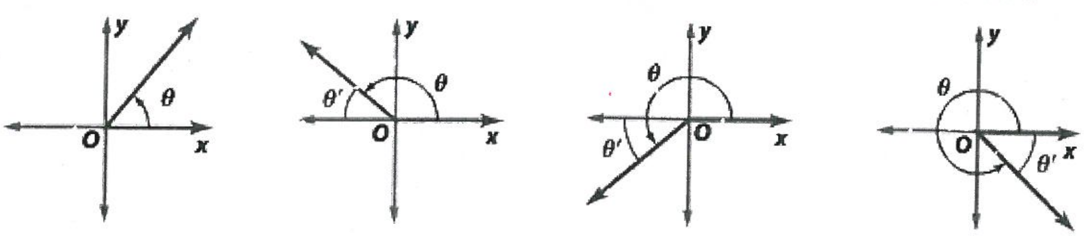
$\theta' = \pi - \theta$

$\theta' = \pi - \frac{3\pi}{4}$

$\theta' = \frac{\pi}{4}$

KeyConcept Reference Angle Rules

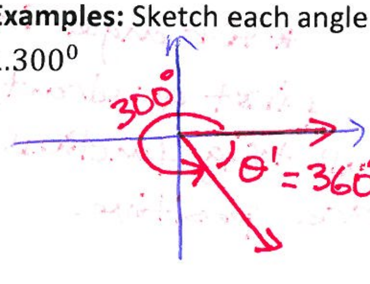
If θ is an angle in standard position, its reference angle θ' is the acute angle formed by the terminal side of θ and the *x*-axis. The reference angle θ' for any angle θ, $0° < \theta < 360°$ or $0 < \theta < 2\pi$, is defined as follows.

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
			
$\theta' = \theta$	$\theta' = 180° - \theta$ $\theta' = \pi - \theta$	$\theta' = \theta - 180°$ $\theta' = \theta - \pi$	$\theta' = 360° - \theta$ $\theta' = 2\pi - \theta$

To find a reference angle for angles outside the interval $0° < \theta < 360°$ or $0 < \theta < 2\pi$, first find a corresponding coterminal angle in this interval.

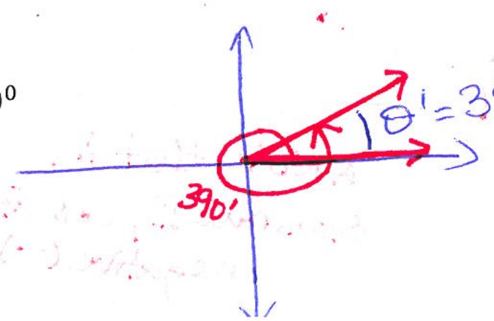
Examples: Sketch each angle and find its reference angle.

1. 300^0



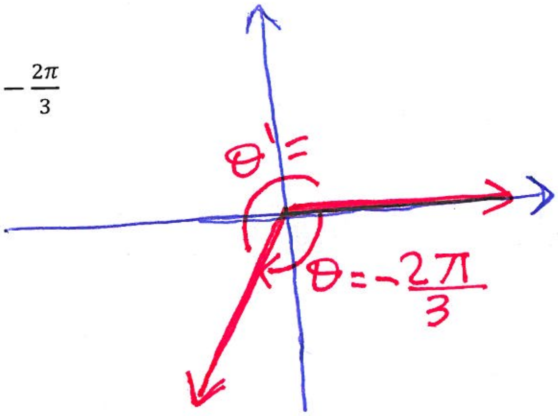
$\theta' = 360° - 300 = 60°$

2. 390^0

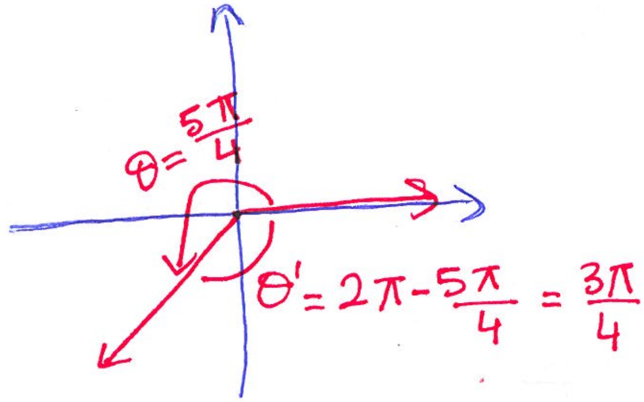


$\theta' = 390° - 360° = 30°$

3. $-\frac{2\pi}{3}$



4. $\frac{5\pi}{4}$



Finding Trigonometric Values Using the Unit Circle:

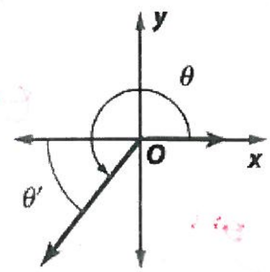
Because the trigonometric values of an angle and its reference angle are equal or differ only in sign, you can use the following steps to find the value of a trigonometric function of any angle θ.

KeyConcept Evaluating Trigonometric Functions of Any Angle

Step 1 Find the reference angle θ'.

Step 2 Find the value of the trigonometric function for θ'.

Step 3 Using the quadrant in which the terminal side of θ lies, determine the sign of the trigonometric function value of θ.



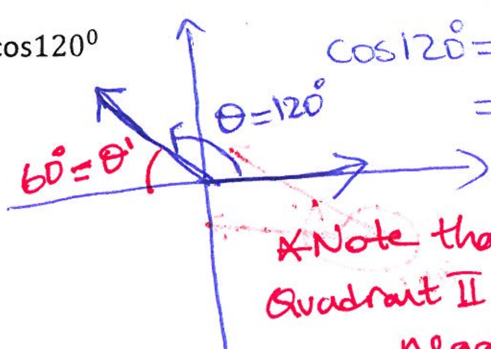
You can easily see from the table below.

Quadrant II	Quadrant I
$\sin\theta$: +	$\sin\theta$: +
$\cos\theta$: −	$\cos\theta$: +
$\tan\theta$: −	$\tan\theta$: +
Quadrant III	**Quadrant IV**
$\sin\theta$: −	$\sin\theta$: −
$\cos\theta$: −	$\cos\theta$: +
$\tan\theta$: +	$\tan\theta$: −

θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Examples: Find the exact values of each expression. If it is undefined, write undefined.

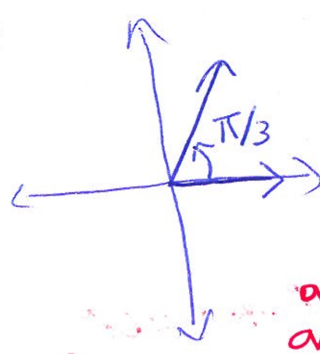
1. $\cos 120^0$



$\cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}$

*Note that in Quadrant II, cos is negative (−)

2. $\sin\frac{\pi}{3}$

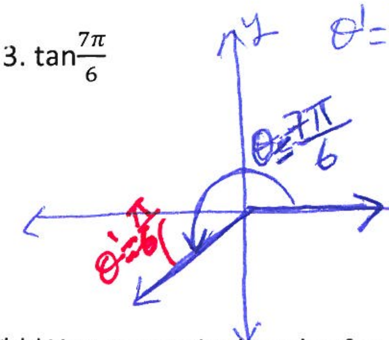


$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

* For angles in first quadrant, all trig functions are positive

Terminal side of θ is in 3rd Quadrant

3. $\tan\frac{7\pi}{6}$



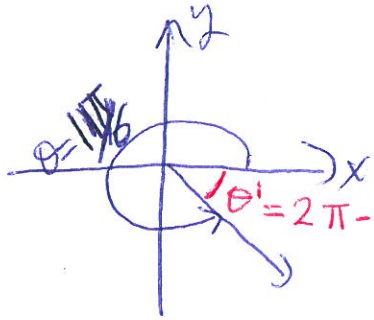
$\theta' = \frac{7\pi}{6} - \pi = \frac{\pi}{6}$

Then, in 3rd quadrant θ is positive

$\tan\frac{7\pi}{6} = \tan\frac{\pi}{6}$

$= \frac{\sqrt{3}}{3}$

4. $\csc\frac{11\pi}{6}$



$\theta' = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$

$\csc\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}}$

$= \frac{1}{-1/2}$

$= -2$

***Use coterminal angles for given angles that are not on the unit circle.

5. $\cos\frac{11\pi}{4}$

$\frac{11\pi}{4} - 2\pi = \frac{3\pi}{4}$

$\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

$\cos\frac{11\pi}{4} = -\frac{\sqrt{2}}{2}$

6. $tan\frac{19\pi}{6}$

$\frac{19\pi}{6} - 3\pi = \frac{\pi}{6}$

$\tan\frac{\pi}{6} = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$

$= \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

* Once you memorize the values from table, you can directly write answer.

BONUS:

1. $\tan 270^0$

$\tan 270° = \frac{y}{x}$

$\tan 270° = \frac{-1}{0}$

$\tan 270° =$ undefined

2. $\sin(\frac{-2\pi}{3})$

$-\frac{2\pi}{3} + \pi = \frac{\pi}{3}$

$\sin\left(-\frac{2\pi}{3}\right) = \sin\frac{\pi}{3}$

$= \frac{\sqrt{3}}{2}$

Use One Trigonometric Value to Find Others:

Examples: Find the exact values of the five remaining trigonometric functions of θ.

1. $tan\theta = \frac{5}{12}$, where $sin\theta < 0$

$\sin\theta = \frac{y}{r}$, ***So, in order for $\sin\theta$ to be less than zero, y must be negative. Only 3rd quadrant $\tan(+)$, $\sin(-)$

$\tan\theta = \frac{x}{y}$

So, then,

If $y = -5$, then $x = -12$ since $\tan\theta > 0$, $r = \sqrt{5^2 + 12^2} = 13$

$\sin\theta = \frac{-5}{13}$ //, $\csc\theta = \frac{-13}{5}$ // $\cot\theta = \frac{12}{5}$ //

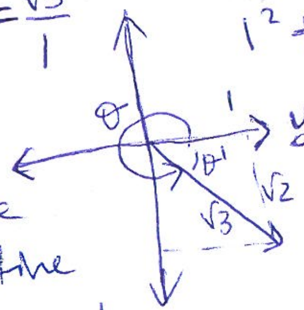
$\cos\theta = \frac{-12}{13}$ // $\sec\theta = \frac{-13}{12}$ //

2. $sec\theta = \sqrt{3}, tan\theta < 0$

$\tan\theta = \frac{y}{1} < 0$ *** So, y must be negative

$\sec\theta = \frac{r}{x} = \frac{\sqrt{3}}{1}$

$\sec\theta = \frac{1}{\cos\theta}$



$1^2 + y^2 = (\sqrt{3})^2$

$y = \sqrt{2}$

cos is positive
tan is negative
only in 4^{th} quadralt

$\sin\theta = -\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{-\sqrt{6}}{3}$

$\cos\theta = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ //

$\tan\theta = \frac{-\sqrt{2}}{1} = -\sqrt{2}$

$\csc\theta = \frac{-\sqrt{6}}{2}$

$\cot\theta = -\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$

Homework: Complete All Questions Due 2/18/2015

Trigonometric Functions on the Unit Circle

The given point lies on the terminal side of an angle *θ* in standard position. Find the values of the six trigonometric functions of *θ*.

1. $(-1, 5)$

2. $(7, 0)$

3. $(-3, -4)$

4. $(1, -2)$

5. $(-3, 1)$

6. $(2, -4)$

Sketch each angle. Then find its reference angle.

7. 330°

8. $-\frac{3\pi}{4}$

9. $\frac{7\pi}{6}$

10. $\frac{7\pi}{4}$

11. 135°

12. $-\frac{\pi}{3}$

Find the exact value of each expression. If undefined, write *undefined*.

13. $\csc 90°$

14. $\tan 270°$

15. $\sin(-90°)$

16. $\cos \frac{3\pi}{2}$

17. $\sec\left(-\frac{\pi}{4}\right)$

18. $\cot \frac{5\pi}{6}$