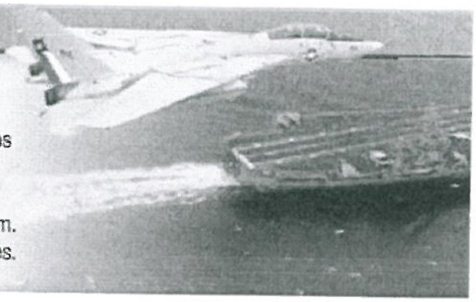
Hand

5.1 Trigonometric Identities

Then	Now	Why?
You found trigonometric values using the unit circle. (Lesson 4-3)	1 Identify and use basic trigonometric identities to find trigonometric values. 2 Use basic trigonometric identities to simplify and rewrite trigonometric expressions.	Many physics and engineering applications, such as determining the path of an aircraft, involve trigonometric functions. These functions are made more flexible if you can change the trigonometric expressions involved from one form to an equivalent but more convenient form. You can do this by using trigonometric identities.



Objectives: I can identify and use basic trigonometric identities to find trigonometric values. I can use basic trigonometric identities to simplify and rewrite trigonometric expressions.

New Vocabulary

Identity: an equation is an identity if the left side is equal to the right side for all values of the variable for which both sides are defined. $\frac{x^2-9}{x-3} = x+3$ both sides of the equation are defined and equal for all x such that $x \neq 3$

Trigonometric identity: an equation that involve trigonometric functions.

Using the following basic trigonometric identities, you can find trigonometric values. As with any fraction, denominator cannot be equal to zero.

KeyConcept Reciprocal and Quotient Identities

Reciprocal Identities			Quotient Identities
$\sin\theta = \frac{1}{\csc\theta}$	$\cos\theta = \frac{1}{\sec\theta}$	$\tan\theta = \frac{1}{\cot\theta}$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\csc\theta = \frac{1}{\sin\theta}$	$\sec\theta = \frac{1}{\cos\theta}$	$\cot\theta = \frac{1}{\tan\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$

Examples:

1. If $\csc\theta = \frac{7}{4}$, find $\sin\theta$.

$\csc\theta = \frac{1}{\sin\theta}$

switch $\sin\theta$ & $\csc\theta$, then

$\sin\theta = \frac{1}{\csc\theta}$

$\sin\theta = \frac{1}{\frac{7}{4}}$

$\boxed{\sin\theta = \frac{4}{7}}$

2. If $\sec x = \frac{5}{3}$, find $\cos x$.

$\sec x = \frac{1}{\cos x}$

Then, $\cos x = \frac{1}{\sec x}$

$\cos x = \frac{1}{\frac{5}{3}}$

$\boxed{\cos x = \frac{3}{5}}$

3. If $\cot x = \frac{2}{5\sqrt{5}}$ and $\sin x = \frac{\sqrt{5}}{3}$, find $\cos x$.

$\frac{\cot x}{1} = \frac{\cos x}{\sin x}$

$\cos x = (\cot x)\times(\sin x)$

$\cos x = \left(\frac{2}{5\sqrt{5}}\right)\left(\frac{\sqrt{5}}{3}\right)$

$\cos x = \frac{2\cancel{\sqrt{5}}}{15\cancel{\sqrt{5}}}$

$\boxed{\cos x = \frac{2}{15}}$

4. If $\csc \beta = \frac{25}{7}$ and $\sec \beta = \frac{25}{24}$, find $\tan \beta$.

$\sin \beta = \frac{1}{\csc \beta}$ & $\cos \beta = \frac{1}{\sec \beta}$

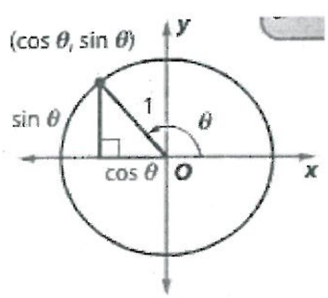
$\sin \beta = \frac{1}{\frac{25}{7}} = \frac{7}{25}$, $\cos \beta = \frac{1}{\frac{25}{24}} = \frac{24}{25}$

$\tan \beta = \frac{\sin \beta}{\cos \beta}$

$\tan \beta = \frac{7/25}{24/25}$

$\tan \beta = \frac{7}{\cancel{25}} \times \frac{\cancel{25}}{24}$

$\boxed{\tan \beta = \frac{7}{24}}$



Recall from Lesson 4-3 that trigonometric functions can be defined on a unit circle as shown. Notice that for any angle θ, sine and cosine are the directed lengths of the legs of a right triangle with hypotenuse 1. We can apply the Pythagorean Theorem to this right triangle to establish another basic trigonometric identity.

$(\sin \theta)^2 + (\cos \theta)^2 = 1^2$ Pythagorean Theorem

$\sin^2 \theta + \cos^2 \theta = 1$ Simplify.

While the signs of these directed lengths may change depending on the quadrant in which the triangle lies, notice that because these lengths are squared, the equation above holds true for any value of θ. This equation is one of three **Pythagorean identities**.

ReadingMath

Powers of Trigonometric Functions $\sin^2 \theta$ is read as *sine squared theta* and interpreted as the square of the quantity $\sin \theta$.

KeyConcept Pythagorean Identities

$\sin^2 \theta + \cos^2 \theta = 1$ $\tan^2 \theta + 1 = \sec^2 \theta$ $\cot^2 \theta + 1 = \csc^2 \theta$

Notice the shorthand notation used to represent powers of trigonometric functions: $\sin^2 \theta = (\sin \theta)^2$, $\cos^2 \theta = (\cos \theta)^2$, $\tan^2 \theta = (\tan \theta)^2$, and so on.

Examples: *Find the value of each expression using the given information.*

1. $\sin\theta$ and $\cos\theta$; $\tan\theta = -8$ and $\sin\theta > 0$ *Use the Pythagorean Identity that involve $\tan\theta$.

$\tan^2\theta + 1 = \sec^2\theta$

$(-8)^2 + 1 = \sec^2\theta$

$65 = \sec^2\theta$

$\pm\sqrt{65} = \sec\theta$

*Since tan is negative and sin must be positive, then cos must be negative.

$\boxed{\sec\theta = -\sqrt{65}}$

$\cos\theta = \frac{1}{\sec\theta}$

So, $\boxed{\cos\theta = \frac{1}{-\sqrt{65}} = \frac{-\sqrt{65}}{65}}$

Now, since we know $\tan\theta$ and $\cos\theta$, we can find $\sin\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\frac{-8}{1} = \frac{\sin\theta}{\frac{-\sqrt{65}}{65}}$

$\sin\theta = (-8)\left(-\frac{\sqrt{65}}{65}\right)$

$\boxed{\sin\theta = \frac{8\sqrt{65}}{65}}$

2. $\csc\theta$ and $\tan\theta$; $\cot\theta = -3$, $\cos\theta < 0$

$\cot^2\theta + 1 = \csc^2\theta$

$(-3)^2 + 1 = \csc^2\theta$

$10 = \csc^2\theta$

$\pm\sqrt{10} = \csc\theta$

* $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\cos\theta$ is negative when $\cot\theta$ is negative. So, $\sin\theta$ must be positive. $\sin\theta = \frac{1}{\csc\theta}$, then $\csc\theta$ must be positive.

Then, $\boxed{\csc\theta = \sqrt{10}}$

$\cot\theta = -3$, $\tan\theta = \frac{1}{\cot\theta}$

$\boxed{\tan\theta = \frac{-1}{3}}$

3. $\cot x$ and $\sec x$; $\sin x = \frac{1}{6}$, $\cos x > 0$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$\cos^2\theta = 1 - \left(\frac{1}{6}\right)^2$

$\cos^2\theta = \frac{1}{1}_{(36)} - \frac{1}{36}$

$\cos^2\theta = \frac{35}{36}$

$\cos\theta = \pm\frac{\sqrt{35}}{6}$

* cos x is positive, so,

$\boxed{\cos\theta = \frac{\sqrt{35}}{6}}$

$\cot x = \frac{\cos x}{\sin x} = \frac{\sqrt{35}/6}{1/6}$

$\boxed{\cot x = \sqrt{35}}$

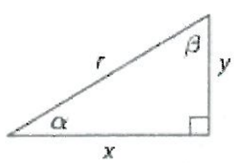
$\sec x = \frac{1}{\cos x} = \frac{1}{\frac{\sqrt{35}}{6}}$

$\sec x = \frac{6}{\sqrt{35}} \times \left(\frac{\sqrt{35}}{\sqrt{35}}\right)$

$\boxed{\sec x = \frac{6\sqrt{35}}{35}}$

***Cofunction: You can use cofunction and odd-even identities to find trigonometric values

Another set of basic trigonometric identities involve cofunctions. A trigonometric function f is a **cofunction** of another trigonometric function g if $f(\alpha) = g(\beta)$ when α and β are complementary angles. In the right triangle shown, angles α and β are complementary angles. Using the right triangle ratios, you can show that the following statements are true.



$$\sin\alpha = \cos\beta = \cos(90° - \alpha) = \frac{y}{r}$$

$$\tan\alpha = \cot\beta = \cot(90° - \alpha) = \frac{y}{x}$$

$$\sec\alpha = \csc\beta = \csc(90° - \alpha) = \frac{r}{y}$$

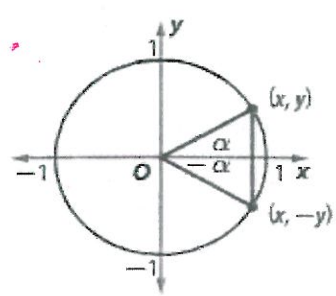
StudyTip

Writing Cofunction Identities Each of the cofunction identities can also be written in terms of degrees. For example, $\sin\theta = \cos(90° - \theta)$.

KeyConcept Cofunction Identities

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$	$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$
$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

You have also seen that each of the basic trigonometric functions—sine, cosine, tangent, cosecant, secant, and cotangent—is either odd or even. Using the unit circle, you can show that the following statements are true.



$\sin\alpha = y$ $\quad$ $\sin(-\alpha) = -y$

$\cos\alpha = x$ $\quad$ $\cos(-\alpha) = x$

Recall from Lesson 1-2 that a function f is even if for every x in the domain of f, $f(-x) = f(x)$ and odd if for every x in the domain of f, $f(-x) = -f(x)$. These relationships lead to the following odd-even identities.

*All Students Take Calculus
$+$ $\quad$ $\sin +$ $\quad$ $\tan +$ $\quad$ $\cos +$

KeyConcept Odd-Even Identities

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\csc(-\theta) = -\csc\theta$	$\sec(-\theta) = \sec\theta$	$\cot(-\theta) = -\cot\theta$

Examples:

1. If $\tan\theta = 1.28$, find $\cot\left(\theta - \frac{\pi}{2}\right)$

odd-even identity

$$\cot\left[-\left(\frac{\pi}{2}-\theta\right)\right] = -\cot\left(\frac{\pi}{2}-\theta\right)$$
$$= -\tan\theta$$
$$= -(1.28)$$
$$\cot\left[-\left(\frac{\pi}{2}-\theta\right)\right] = -1.28$$

2. If $\sin x = -0.37$, find $\cos\left(x - \frac{\pi}{2}\right)$

odd-even identity

$$\cos\left(x-\frac{\pi}{2}\right) = \cos\left[-\left(\frac{\pi}{2}-x\right)\right]$$
$$= \cos\left(\frac{\pi}{2}-x\right)$$
$$= \sin\theta$$
$$= -0.37$$

2 Simplify and Rewrite Trigonometric Expressions

To simplify a trigonometric expression, start by rewriting it in terms of one trigonometric function or in terms of sine and cosine only.

Simplify and Rewrite Trigonometric Expressions:

Examples: Simplify

1. $\csc\theta\sec\theta - \cot\theta$ $\quad$ *Use Reciprocal & Quotient Identities

$$\csc\theta\cdot\sec\theta - \cot\theta = \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta} - \frac{\cos\theta}{\sin\theta}$$
$$= \frac{1}{\sin\theta\cdot\cos\theta} - \frac{\cos\theta}{\sin\theta}$$ Find LCD to subtract fractions.
$$= \frac{1}{\sin\theta\cdot\cos\theta} - \frac{\cos^2\theta}{\sin\theta\cdot\cos\theta}$$
$$= \frac{1-\cos^2\theta}{\sin\theta\cdot\cos\theta}$$ Pythagorean Identity
$$= \frac{\cancel{\sin^2}\theta}{\cancel{\sin}\theta\cdot\cos\theta} = \frac{\sin\theta}{\cos\theta} = \boxed{\tan\theta}$$

2. $\sec x - \tan x \sin x = \frac{1}{\cos x} - \frac{\sin x}{\cos x} \cdot \sin x$ → Reciprocal and Quotient Identities

$= \frac{1}{\cos x} - \frac{\sin^2 x}{\cos x}$ → Subtract fractions with same LCD

$= \frac{1-\sin^2 x}{\cos x}$ → Pythagorean Identity

$= \frac{\cos^2 x}{\cos x}$

$= \cos x$

3. $\sin^2 x \cos x - \left(\sin\frac{\pi}{2} - x\right)$ → use cofunction identity

$= \sin^2 x \cos x - \cos x$

$= \cos x(\sin^2 x - 1)$ → Factor (GCF)

$= \cos x(-\cos^2 x)$

$= -\cos^3 x$

4. $-\csc\left(\frac{\pi}{2} - x\right) - \tan^2 x \sec x = -\sec x - \tan^2 x \sec x$ ← cofunction Identity

$= -\sec x - \tan^2 x \sec x$

$= -\sec x(1 + \tan^2 x)$ → Pythagorean Identity

$= -\sec x(\sec^2 x)$

$= -\sec^3 x$

5. $\frac{\sin x \cos x}{1-\sin x} - \frac{1+\sin x}{\cos x} = \frac{\sin x \cos^2 x}{(1-\sin x)\cos x} - \frac{(1-\sin x)(1+\sin x)}{\cos x(1-\sin x)}$ LCD

$(\cos x)$ $(1-\sin x)$

$= \frac{\sin x \cos^2 x - (1-\sin^2 x)}{(1-\sin x)\cos x}$ combine fractions & simplify numerator

$= \frac{\sin x \cos^2 x - \cos^2 x}{\cos x(1-\sin x)}$

$= \frac{\cos^2 x(\sin x - 1)}{\cos x(1-\sin x)}$ Factor out GCF

$= \frac{\cos^2 x(-1)(1-\sin x)}{\cos x(1-\sin x)}$ simplify

$= \frac{-\cos x(1-\sin x)}{(1-\sin x)}$ Simplify

$= -\cos x$

6. $\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} = \frac{\cos^2 x + (1+\sin x)(1+\sin x)}{\cos x(1+\sin x)}$ LCD

$(\cos x)$ $(1+\sin x)$

$= \frac{\cos^2 x + 1 + 2\sin x + \sin^2 x}{\cos x(1+\sin x)}$

$= \frac{\cos^2 x + 1 + \sin^2 x + 2\sin x}{\cos x(1+\sin x)}$ Rearrange the numerator

$= \frac{\cos^2 x + \sin^2 x + 1 + 2\sin x}{\cos x(1+\sin x)}$ Pythagorean Identity

$= \frac{1+1+2\sin x}{\cos x(1+\sin x)}$

$= \frac{2+2\sin x}{\cos x(1+\sin x)}$

$= \frac{2(1+\sin x)}{\cos x(1+\sin x)} = \frac{2}{\cos x} = 2\sec x$

Factor GCF

Reciprocal identity

Rewrite to Eliminate Fractions:

Examples: Rewrite each expression as an expression that does not involve a fraction.

1. $\frac{1}{1+\cos x}$

*multiply by denominator's conjugate

$$\frac{1}{1+\cos x}\cdot\frac{(1-\cos x)}{(1-\cos x)} = \frac{1-\cos x}{1-\cos^2 x}$$

$$= \frac{1-\cos x}{\sin^2 x}$$

*multiply by denominator's reciprocal

$$= \frac{(1-\cos x)}{\sin^2 x}\cdot\frac{(\csc^2 x)}{(\csc^2 x)}$$

simplify

$$= \frac{\csc^2 x - \cos x \csc^2 x}{\cancel{\sin^2 x}\cdot\frac{1}{\sin^2 x}}$$

$$= \csc^2 x - \cos x \cdot \csc^2 x$$

$$= \csc^2 x - \cos x \cdot \frac{1}{\sin x}\csc x$$

$$= \csc^2 x - \frac{\cos x}{\sin x}\cdot \csc x$$

$$= \csc^2 x - \cot x \csc x$$

2. $\frac{\cos^2 x}{1-\sin x}$

$$\frac{\cos^2 x}{1-\sin x}\cdot\frac{(1+\sin x)}{(1+\sin x)} = \frac{\cos^2 x + \cos^2 x \sin x}{1-\sin^2 x}$$

$$= \frac{\cos^2 x + \cos^2 x \sin x}{1-\sin^2 x}$$

$$= \frac{\cos^2 x (1+\sin x)}{\cos^2 x}$$

$$= \frac{\cancel{\cos^2 x}(1+\sin x)}{\cancel{\cos^2 x}}$$

$$= 1+\sin x$$